MINI-STATISTICS (PH.D.) | SUMMER 2026

Transformations of Random Variables

Selected lecture handout | Support, distribution functions, and changes of scale

A transformation changes the values a random variable can take. To find the new distribution, start with its *support*, then work with the cumulative distribution function (CDF). The density follows by differentiation.

1. A change of scale

Suppose X has density

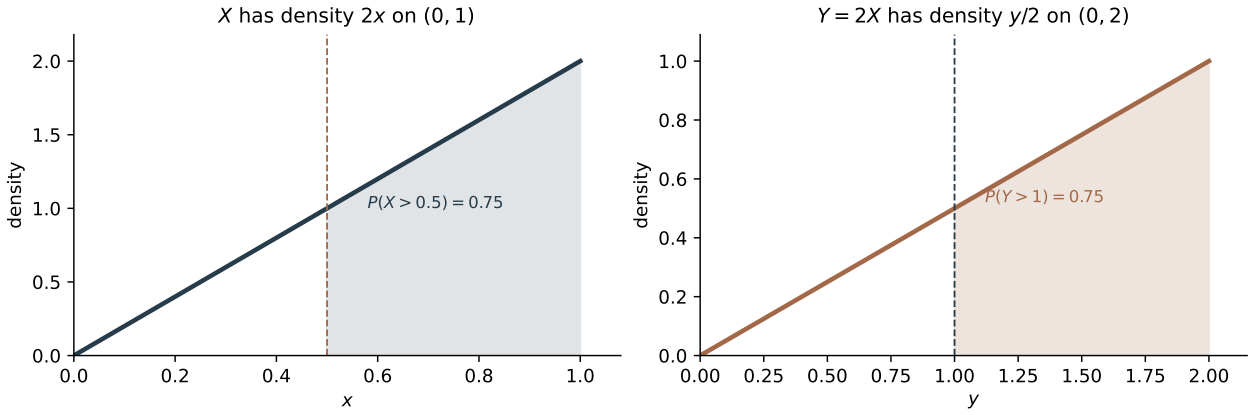
$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{and let} \qquad Y = 2X.$$

The support of Y is $(0, 2)$. For $0 < y < 2$,

$$F_Y(y) = P(2X \le y) = P\Big(X \le \frac{y}{2}\Big) = \int_0^{y/2} 2x\, dx = \frac{y^2}{4}.$$

Therefore

$$f_Y(y) = \begin{cases} \dfrac{y}{2}, & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$



Check. The horizontal scale doubles, so the density must change as well. Both descriptions assign the same probability to the same event:

$$P(X > 1/2) = \int_{1/2}^{1} 2x\, dx = \frac{3}{4} = \int_1^2 \frac{y}{2}\, dy = P(Y > 1).$$

2. A many-to-one transformation

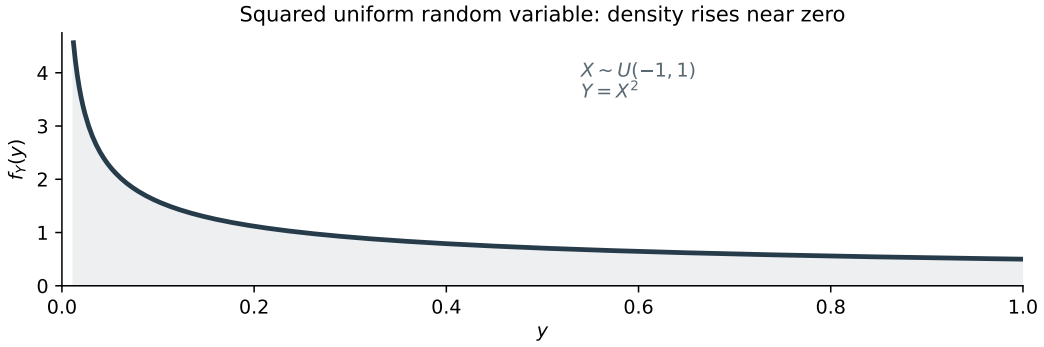
Now let $X \sim U(-1, 1)$ and $Y = X^2$. Two values of X give the same positive value of Y: $+\sqrt{y}$ and $-\sqrt{y}$. Using only the positive branch would miss half the probability.

For $0 < y < 1$,

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \frac{2\sqrt{y}}{2} = \sqrt{y}.$$

Differentiating gives

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$



The density is high near zero because squaring maps both small positive and small negative values of X close to zero. The density becomes unbounded as $y \downarrow 0$, but its total area is still one:

$$\int_0^1 \frac{1}{2\sqrt{y}}\, dy = 1.$$

Compare the examples. In the first example, multiplying by two stretches the support. In the second, squaring folds negative and positive values together. Both transformations change the density, but for different reasons.

3. Questions for practice

1. Let $X \sim U(0, 1)$ and $Y = -2 \log X$. Find the support of Y and derive its CDF. Remember that dividing an inequality by a negative number reverses its direction.
2. In the first example, replace $Y = 2X$ with $Y = 8X^3$. Which step of the argument changes? Derive the new density and verify that it integrates to one.

Adapted from the 2026 Mini-Statistics review course. The central teaching approach is to derive the distribution from the CDF, then explain how the transformation changes the support and density.